

SOME SYMMETRIC q -CONGRUENCES

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ABSTRACT. We prove some symmetric q -congruences.

1. INTRODUCTION

Suppose that f_0, f_1, f_2, \dots are a sequence of integers. Let

$$\hat{f}_k(q) = \sum_{j=0}^k (-1)^j \binom{k}{j} f_j.$$

In [2, Theorem 2.4], Sun proved the following symmetric congruence:

$$\sum_{k=0}^{p-1} \binom{\alpha}{k} \binom{-1-\alpha}{k} f_k \equiv (-1)^{\langle \alpha \rangle_p} \sum_{k=0}^{p-1} \binom{\alpha}{k} \binom{-1-\alpha}{k} \hat{f}_k \pmod{p^2}, \quad (1.1)$$

where p is an odd prime, $\alpha \in \mathbb{Q}$ is p -integral and $\langle \alpha \rangle_p$ is the least non-negative residue of α modulo p . With the help of (1.1), Sun obtained many interesting congruences modulo p .

In this paper, we shall give a q -analogue of (1.1). Define

$$(x; q)_n = \begin{cases} (1-x)(1-xq) \cdots (1-xq^{n-1}), & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases}$$

Define the q -binomial coefficient

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{\alpha-k+1}; q)_k}{(q; q)_k}.$$

In particular, we set $\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = 0$ if $k < 0$. Suppose that $f_1(q), f_2(q), \dots$ are a sequence of polynomials in q with integral coefficients. Let

$$\hat{f}_k(q) = \sum_{j=0}^k (-1)^j q^{\binom{j+1}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q f_j(q)$$

for $0 \leq k \leq n$.

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Theorem 1.1. *Let $n \geq 2$ and $d \geq 1$ with $(n, d) = 1$. Suppose that $f_1(q), \dots, f_n(q) \in \mathbb{Z}[q]$ and r is an integer. Then if n is odd,*

$$\begin{aligned} & q^{d\binom{a+1}{2} + \frac{(ad+r)(n-1-2a)}{2}} \sum_{k=0}^{n-1} \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k}{(q^d; q^d)_k^2} \cdot q^{dk} f_k(q^d) \\ & \equiv (-1)^a \sum_{k=0}^{n-1} \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k}{(q^d; q^d)_k^2} \cdot q^{dk} \hat{f}_k(q^d) \pmod{\Phi_n(q)^2}, \end{aligned} \quad (1.2)$$

where $a = \langle -r/d \rangle_n$ and Φ_n is the n -th cyclotomic polynomial. And when n is even,

$$\begin{aligned} & q^{d\binom{a+1}{2} + \frac{(ad+r)(n-1-2a)}{2}} \sum_{k=0}^{n-1} \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k}{(q^d; q^d)_k^2} \cdot q^{dk} f_k(q^d) \\ & \equiv (-1)^{a+\frac{ad+r}{n}} \sum_{k=0}^{n-1} \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k}{(q^d; q^d)_k^2} \cdot q^{dk} \hat{f}_k(q^d) \pmod{\Phi_n(q)^2}. \end{aligned} \quad (1.3)$$

Now for odd $n \geq 3$, replacing q by q^{-1} in (1.4) and noting that

$$\Phi_n(q^{-1}) = q^{-\phi(n)} \Phi_n(q)$$

where ϕ is the Euler totient function, we get

$$\begin{aligned} & q^{-d\binom{a+1}{2} - \frac{(ad+r)(n-1-2a)}{2}} \sum_{k=0}^{n-1} \frac{(q^{-r}; q^{-d})_k (q^{r-d}; q^{-d})_k}{(q^{-d}; q^{-d})_k^2} \cdot q^{-dk} f_k(q^{-d}) \\ & \equiv (-1)^a \sum_{k=0}^{n-1} \frac{(q^{-r}; q^{-d})_k (q^{r-d}; q^{-d})_k}{(q^{-d}; q^{-d})_k^2} \cdot q^{-dk} \hat{f}_k(q^{-d}) \pmod{\Phi_n(q)^2}. \end{aligned}$$

Notice that

$$\hat{f}_k(q^{-1}) = \sum_{j=0}^k (-1)^j q^{-\binom{j+1}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^{-1}} f_j(q^{-1}) = \sum_{j=0}^k (-1)^j q^{\binom{j}{2} - kj} \begin{bmatrix} k \\ j \end{bmatrix}_q f_j(q^{-1})$$

and

$$\frac{(q^{-r}; q^{-d})_k (q^{r-d}; q^{-d})_k}{(q^{-d}; q^{-d})_k^2} = q^{dk} \cdot \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k}{(q^d; q^d)_k^2}.$$

We have

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k}{(q^d; q^d)_k^2} \cdot g_k(q^d) \\ & \equiv (-1)^a q^{d\binom{a+1}{2} + \frac{(ad+r)(n-1-2a)}{2}} \sum_{k=0}^{n-1} \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k}{(q^d; q^d)_k^2} \cdot \tilde{g}_k(q^d) \pmod{\Phi_n(q)^2}, \end{aligned}$$

where

$$\tilde{g}_k(q) = \sum_{j=0}^k (-1)^j q^{\binom{j}{2} - kj} \begin{bmatrix} k \\ j \end{bmatrix}_q g_j(q).$$

The similar discussion is also valid when n is even. Thus

Theorem 1.2. *Let $n \geq 2$ and $d \geq 1$ with $(n, d) = 1$. Suppose that $f_1(q), \dots, f_n(q) \in \mathbb{Z}[q]$ and r is an integer. Then if n is odd,*

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k}{(q^d; q^d)_k^2} \cdot f_k(q^d) \\ & \equiv (-1)^a q^{d\binom{a+1}{2} + \frac{(ad+r)(n-1-2a)}{2}} \sum_{k=0}^{n-1} \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k}{(q^d; q^d)_k^2} \cdot \tilde{f}_k(q^d) \pmod{\Phi_n(q)^2}, \end{aligned} \quad (1.4)$$

where $a = \langle -r/d \rangle_n$ and Φ_n is the n -th cyclotomic polynomial. And when n is even,

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k}{(q^d; q^d)_k^2} \cdot f_k(q^d) \\ & \equiv (-1)^{a + \frac{ad+r}{n}} q^{d\binom{a+1}{2} + \frac{(ad+r)(n-1-2a)}{2}} \sum_{k=0}^{n-1} \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k}{(q^d; q^d)_k^2} \cdot \tilde{f}_k(q^d) \pmod{\Phi_n(q)^2}. \end{aligned} \quad (1.5)$$

2. PROOF OF THEOREM 1.1

Below we slightly extend the notion of congruence. Let

$$\mathfrak{X}(q) = \left\{ \sum_{j=1}^s c_j q^{\beta_j} : s \geq 1, c_1, \dots, c_s \in \mathbb{C}, \beta_1, \dots, \beta_s \in \mathbb{R} \right\}.$$

For $f(q), g(q), h(q) \in \mathfrak{X}(q)$, we say

$$f(q) \equiv g(q) \pmod{h(q)},$$

provided

$$\frac{f(q) - g(q)}{h(q)} \in \mathfrak{X}(q).$$

Theorem 2.1. *Suppose that $n \geq 2$, $\alpha \in \mathbb{Q}$ and the denominator of α is prime to n . Then when n is odd,*

$$\begin{aligned} & (-1)^a q^{\binom{a+1}{2} + sna - s\binom{n}{2}} \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q \begin{bmatrix} -1 - \alpha \\ k \end{bmatrix}_q f_k(q) \\ & \equiv \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q \begin{bmatrix} -1 - \alpha \\ k \end{bmatrix}_q \hat{f}_k(q) \pmod{\Phi_n(q)^2}, \end{aligned}$$

where $a = \langle \alpha \rangle_n$ and $s = (\alpha - a)/n$. And if n is even, then

$$\begin{aligned} & (-1)^{a+s} q^{\binom{a+1}{2} + sna - s \binom{n}{2}} \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q \begin{bmatrix} -1 - \alpha \\ k \end{bmatrix}_q f_k(q) \\ & \equiv \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q \begin{bmatrix} -1 - \alpha \\ k \end{bmatrix}_q \hat{f}_k(q) \pmod{\Phi_n(q)^2}. \end{aligned}$$

Let

$$\hat{f}_k(q) = \sum_{i=0}^k (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q f_i(q).$$

Suppose that p is an odd prime and α are positive integer, $0 \leq \alpha < p \leq p-1$. Let $a = \langle \alpha \rangle_n$. Then

Proof. Using the q -Chu-Vandemonde identity, we have

$$\begin{aligned} \begin{bmatrix} a + sn \\ k \end{bmatrix}_q &= \sum_{j=0}^k q^{(sn-j)(k-j)} \begin{bmatrix} sn \\ j \end{bmatrix}_q \begin{bmatrix} a \\ k-j \end{bmatrix}_q \\ &\equiv q^{snk} \begin{bmatrix} a \\ k \end{bmatrix}_q + \sum_{j=1}^k q^{-j(k-j)} \begin{bmatrix} sn \\ j \end{bmatrix}_q \begin{bmatrix} a \\ k-j \end{bmatrix}_q \pmod{\Phi_n(q)^2}. \end{aligned}$$

Notice that

$$\begin{bmatrix} sn \\ j \end{bmatrix}_q = \frac{[sn]_q}{[j]_q} \begin{bmatrix} sn-1 \\ j-1 \end{bmatrix}_q \equiv 0 \pmod{\Phi_n(q)}$$

and

$$\begin{bmatrix} sn-1 \\ j-1 \end{bmatrix}_q \equiv \begin{bmatrix} -1 \\ j-1 \end{bmatrix}_q = (-1)^{j-1} q^{-\binom{j}{2}} \pmod{\Phi_n(q)}.$$

So

$$\begin{bmatrix} a + sn \\ k \end{bmatrix}_q \equiv q^{snk} \begin{bmatrix} a \\ k \end{bmatrix}_q + \sum_{j=1}^k \frac{[sn]_q}{[j]_q} \cdot (-1)^{j-1} q^{-\binom{j}{2} - j(k-j)} \begin{bmatrix} a \\ k-j \end{bmatrix}_q \pmod{\Phi_n(q)^2}.$$

Similarly, we also have

$$\begin{aligned} & \begin{bmatrix} -1 - a - sn \\ k \end{bmatrix}_q \\ & \equiv q^{-snk} \begin{bmatrix} -1 - a \\ k \end{bmatrix}_q + \sum_{j=1}^k \frac{[-sn]_q}{[j]_q} \cdot (-1)^{j-1} q^{-\binom{j}{2} - j(k-j)} \begin{bmatrix} -1 - a \\ k-j \end{bmatrix}_q \pmod{\Phi_n(q)^2}. \end{aligned}$$

Thus

$$\begin{aligned} & \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q \begin{bmatrix} -1-\alpha \\ k \end{bmatrix}_q ((-1)^a q^{\binom{a+1}{2}} f_k(q) - \hat{f}_k(q)) \\ & \equiv \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} -1-a \\ k \end{bmatrix}_q ((-1)^a q^{\binom{a+1}{2}} f_k(q) - \hat{f}_k(q)) \end{aligned} \quad (2.1)$$

$$- s \sum_{k=1}^{n-1} q^{k^2+k} \begin{bmatrix} a \\ k \end{bmatrix}_q f_k(q) \sum_{j=1}^k (-1)^{a+j-1} q^{\binom{a+1}{2} - \binom{j}{2} - j(k-j)} \cdot \frac{[n]_q}{[j]_q} \begin{bmatrix} -1-a \\ k-j \end{bmatrix}_q \quad (2.2)$$

$$+ s \sum_{k=1}^{n-1} q^{k^2+k} \begin{bmatrix} a \\ k \end{bmatrix}_q \hat{f}_k(q) \sum_{j=1}^k (-1)^{j-1} q^{-\binom{j}{2} - j(k-j)} \cdot \frac{[n]_q}{[j]_q} \begin{bmatrix} -1-a \\ k-j \end{bmatrix}_q \quad (2.3)$$

$$+ s \sum_{k=1}^{n-1} q^{k^2+k} \begin{bmatrix} -1-a \\ k \end{bmatrix}_q f_k(q) \sum_{j=1}^k (-1)^{a+j-1} q^{\binom{a+1}{2} - \binom{j}{2} - j(k-j)} \cdot \frac{[n]_q}{[j]_q} \begin{bmatrix} a \\ k-j \end{bmatrix}_q \quad (2.4)$$

$$- s \sum_{k=1}^{n-1} q^{k^2+k} \begin{bmatrix} -1-a \\ k \end{bmatrix}_q \hat{f}_k(q) \sum_{j=1}^k (-1)^{j-1} q^{-\binom{j}{2} - j(k-j)} \cdot \frac{[n]_q}{[j]_q} \begin{bmatrix} a \\ k-j \end{bmatrix}_q \pmod{\Phi_n(q)^2}. \quad (2.5)$$

First, we consider (2.1). Clearly

$$\begin{aligned} & \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} -1-a \\ k \end{bmatrix}_q \hat{f}_k(q) \\ & = \sum_{k=0}^a q^{k^2+k} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} -1-a \\ k \end{bmatrix}_q \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j q^{\binom{j+1}{2}} f_j(q) \\ & = \sum_{j=0}^a (-1)^j q^{\binom{j+1}{2}} f_j(q) \sum_{k=j}^a q^{k^2+k} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} -1-a \\ j \end{bmatrix}_q \begin{bmatrix} -1-a-j \\ k-j \end{bmatrix}_q \\ & = \sum_{j=0}^a (-1)^j q^{\binom{j+1}{2}} f_j(q) \begin{bmatrix} -1-a \\ j \end{bmatrix}_q q^{a^2+a} \sum_{k=j}^a q^{(a-k)(-1-a-k)} \begin{bmatrix} a \\ a-k \end{bmatrix}_q \begin{bmatrix} -1-a-j \\ k-j \end{bmatrix}_q \\ & = \sum_{j=0}^a (-1)^j q^{\binom{j+1}{2}} f_j(q) \begin{bmatrix} -1-a \\ j \end{bmatrix}_q q^{a^2+a} \begin{bmatrix} -1-j \\ a-j \end{bmatrix}_q \\ & = \sum_{j=0}^a (-1)^a q^{j^2+j+\binom{a+1}{2}} f_j(q) \begin{bmatrix} -1-a \\ j \end{bmatrix}_q \begin{bmatrix} a \\ j \end{bmatrix}_q, \end{aligned}$$

where we use the q -Chu-Vandemonde identity in the fourth equality. So (2.1) always vanishes.

Note that

$$\begin{aligned} & \sum_{j=1}^k (-1)^{j-1} q^{-\binom{j}{2} - j(k-j)} \cdot \frac{[n]_q}{[j]_q} \begin{bmatrix} -1-a \\ k-j \end{bmatrix}_q \equiv \sum_{j=1}^k \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q \cdot q^{-j(k-j)} \cdot \frac{[n]_q}{[j]_q} \begin{bmatrix} -1-a \\ k-j \end{bmatrix}_q \\ & \equiv \sum_{j=1}^k q^{(n-j)(k-j)} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} -1-a \\ k-j \end{bmatrix}_q = \begin{bmatrix} n-1-a \\ k \end{bmatrix}_q - q^{nk} \begin{bmatrix} -1-a \\ k \end{bmatrix}_q \pmod{\Phi_n(q)^2}. \end{aligned}$$

Hence

$$(2.2) \equiv -s(-1)^a q^{\binom{a+1}{2}} \sum_{k=1}^{n-1} q^{k^2+k} \begin{bmatrix} a \\ k \end{bmatrix}_q f_k(q) \cdot \left(\begin{bmatrix} n-1-a \\ k \end{bmatrix}_q - q^{nk} \begin{bmatrix} -1-a \\ k \end{bmatrix}_q \right) \quad (2.6)$$

$$\equiv -s(-1)^a q^{\binom{a+1}{2}} \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} n-1-a \\ k \end{bmatrix}_q f_k(q) \quad (2.7)$$

$$+ s(-1)^a q^{\binom{a+1}{2}} \sum_{k=0}^{n-1} q^{k^2+k+nk} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} -1-a \\ k \end{bmatrix}_q f_k(q) \pmod{\Phi_n(q)^2}. \quad (2.8)$$

Similarly, we have

$$\begin{aligned} & \sum_{j=1}^k (-1)^{j-1} q^{-\binom{j}{2} - j(k-j)} \frac{[n]_q}{[j]_q} \begin{bmatrix} a-n \\ k-j \end{bmatrix}_q \\ & \equiv \sum_{j=1}^k q^{(n-j)(k-j)} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} a-n \\ k-j \end{bmatrix}_q = \begin{bmatrix} a \\ k \end{bmatrix}_q - q^{nk} \begin{bmatrix} a-n \\ k \end{bmatrix}_q \pmod{\Phi_n(q)^2}. \end{aligned}$$

So

$$\begin{aligned} (2.4) & \equiv s \sum_{k=1}^{n-1-a} q^{k^2+k+\binom{a+1}{2}} \begin{bmatrix} n-1-a \\ k \end{bmatrix}_q f_k(q) \sum_{j=1}^k (-1)^{a+j-1} q^{-\binom{j}{2} - j(k-j)} \cdot \frac{[n]_q}{[j]_q} \begin{bmatrix} a-n \\ k-j \end{bmatrix}_q \\ & \equiv s(-1)^a q^{\binom{a+1}{2}} \sum_{k=0}^{n-1-a} q^{k^2+k} \begin{bmatrix} n-1-a \\ k \end{bmatrix}_q \begin{bmatrix} a \\ k \end{bmatrix}_q f_k(q) \quad (2.9) \end{aligned}$$

$$- s(-1)^a q^{\binom{a+1}{2}} \sum_{k=0}^{n-1-a} q^{k^2+k+nk} \begin{bmatrix} n-1-a \\ k \end{bmatrix}_q \begin{bmatrix} a-n \\ k \end{bmatrix}_q f_k(q) \pmod{\Phi_p(q)^2}. \quad (2.10)$$

On the other hand, clearly

$$\begin{aligned}
& \sum_{k=1}^{n-1} q^{k^2+k} \begin{bmatrix} a \\ k \end{bmatrix}_q \hat{f}_k(q) \sum_{j=1}^k (-1)^{j-1} q^{-\binom{j}{2}-j(k-j)} \cdot \frac{[n]_q}{[j]_q} \begin{bmatrix} -1-a \\ k-j \end{bmatrix}_q \\
&= \sum_{k=1}^{n-1} q^{k^2+k} \begin{bmatrix} a \\ k \end{bmatrix}_q \sum_{i=0}^k (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q f_i(q) \sum_{j=1}^k (-1)^{j-1} q^{-\binom{j}{2}-j(k-j)} \cdot \frac{[n]_q}{[j]_q} \begin{bmatrix} -1-a \\ k-j \end{bmatrix}_q \\
&= \sum_{i=0}^a (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} a \\ i \end{bmatrix}_q f_i(q) \sum_{j=1}^a \frac{(-1)^{j-1} q^{-\binom{j}{2}} [n]_q}{[j]_q} \sum_{k=j}^a q^{k^2+k-j(k-j)} \begin{bmatrix} a-i \\ a-k \end{bmatrix}_q \begin{bmatrix} -1-a \\ k-j \end{bmatrix}_q \\
&= \sum_{i=0}^a (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} a \\ i \end{bmatrix}_q f_i(q) \sum_{j=1}^a \frac{(-1)^{j-1} q^{-\binom{j}{2}} [n]_q}{[j]_q} \cdot q^{a^2+a+j^2-aj} \begin{bmatrix} -1-i \\ a-j \end{bmatrix}_q,
\end{aligned}$$

where in the last step we use the q -Chu-Vandemonde identity. Thus

$$\begin{aligned}
(2.3) &= s \sum_{i=0}^a (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} a \\ i \end{bmatrix}_q f_i(q) \sum_{j=1}^a \frac{(-1)^{j-1} q^{a^2+a+j^2-aj-\binom{j}{2}} [n]_q}{[j]_q} \begin{bmatrix} -1-i \\ a-j \end{bmatrix}_q \\
&\equiv s \sum_{i=0}^a (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} a \\ i \end{bmatrix}_q f_i(q) \sum_{j=1}^a q^{a^2+a+j^2-aj} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} -1-i \\ a-j \end{bmatrix}_q \\
&\equiv s q^{a^2+a} \sum_{i=0}^a (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} a \\ i \end{bmatrix}_q f_i(q) \sum_{j=1}^a q^{(n-j)(a-j)} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} -1-i \\ a-j \end{bmatrix}_q \\
&= s q^{a^2+a} \sum_{i=0}^a (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} a \\ i \end{bmatrix}_q f_i(q) \cdot \begin{bmatrix} n-1-i \\ a \end{bmatrix}_q \tag{2.11}
\end{aligned}$$

$$- s q^{a^2+a} \sum_{i=0}^a (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} a \\ i \end{bmatrix}_q f_i(q) \cdot q^{na} \begin{bmatrix} -1-i \\ a \end{bmatrix}_q \pmod{\Phi_n(q)^2}. \tag{2.12}$$

Similarly, noting that

$$\begin{aligned}
& \sum_{k=j}^{n-1-a} q^{k^2+k-j(k-j)} \begin{bmatrix} n-1-a-i \\ k-i \end{bmatrix}_q \begin{bmatrix} a \\ k-j \end{bmatrix}_q \\
&\equiv \sum_{k=j}^{n-1-a} q^{a^2+a+j^2+j+aj+(n-1-a-k)(a-k+j)} \begin{bmatrix} n-1-a-i \\ n-1-a-k \end{bmatrix}_q \begin{bmatrix} a \\ k-j \end{bmatrix}_q \\
&= q^{a^2+a+j^2+j+aj} \begin{bmatrix} n-1-i \\ n-1-a-j \end{bmatrix}_q \pmod{\Phi_n(q)},
\end{aligned}$$

we can get

$$\begin{aligned}
(2.5) &\equiv -s \sum_{k=1}^{n-1-a} \begin{bmatrix} n-1-a \\ k \end{bmatrix}_q \hat{f}_k(q) \sum_{j=1}^k \frac{(-1)^{j-1} q^{-\binom{j}{2}} [n]_q}{[j]_q} \cdot q^{k^2+k-j(k-j)} \begin{bmatrix} a \\ k-j \end{bmatrix}_q \\
&\equiv -s \sum_{i=0}^{n-1-a} (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} n-1-a \\ i \end{bmatrix}_q f_i(q) \sum_{j=1}^{n-1-a} \begin{bmatrix} n \\ j \end{bmatrix}_q \cdot q^{a^2+a+j^2+j+aj} \begin{bmatrix} n-1-i \\ n-1-a-j \end{bmatrix}_q \\
&\equiv -s \sum_{i=0}^{n-1-a} (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} n-1-a \\ i \end{bmatrix}_q f_i(q) \sum_{j=1}^{n-1-a} q^{a^2+a+(n-j)(n-1-a-j)} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} -1-i \\ n-1-a-j \end{bmatrix}_q \\
&= -s \sum_{i=0}^{n-1-a} (-1)^i q^{a^2+a+\binom{i+1}{2}} \begin{bmatrix} n-1-a \\ i \end{bmatrix}_q f_i(q) \cdot \begin{bmatrix} n-1-i \\ n-1-a \end{bmatrix}_q \quad (2.13) \\
&+ s \sum_{i=0}^{n-1-a} (-1)^i q^{a^2+a+\binom{i+1}{2}} \begin{bmatrix} n-1-a \\ i \end{bmatrix}_q f_i(q) \cdot q^{n(n-1-a)} \begin{bmatrix} -1-i \\ n-1-a \end{bmatrix}_q \pmod{\Phi_n(q)^2}. \quad (2.14)
\end{aligned}$$

Clearly

$$(2.7) + (2.9) = 0. \quad (2.15)$$

And we also get

$$(2.11) + (2.13) = 0, \quad (2.16)$$

since

$$\begin{aligned}
\begin{bmatrix} a \\ i \end{bmatrix}_q \begin{bmatrix} n-1-i \\ a \end{bmatrix}_q &= \begin{bmatrix} n-1-i \\ i \end{bmatrix}_q \begin{bmatrix} n-1-2i \\ a-i \end{bmatrix}_q \\
&= \begin{bmatrix} n-1-i \\ i \end{bmatrix}_q \begin{bmatrix} n-1-2i \\ n-1-a-i \end{bmatrix}_q = \begin{bmatrix} n-1-a \\ i \end{bmatrix}_q \begin{bmatrix} n-1-i \\ n-1-a \end{bmatrix}_q.
\end{aligned}$$

Noting that

$$(-1)^a q^{ak+\binom{a+1}{2}} \begin{bmatrix} -1-k \\ a \end{bmatrix}_q = \begin{bmatrix} a+k \\ a \end{bmatrix}_q = \begin{bmatrix} a+k \\ k \end{bmatrix}_q = (-1)^k q^{ak+\binom{k+1}{2}} \begin{bmatrix} -1-a \\ k \end{bmatrix}_q,$$

we have

$$(2.8) + (2.12) = s(-1)^a q^{\binom{a+1}{2}} \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} -1-a \\ k \end{bmatrix}_q f_k(q) \cdot (q^{nk} - q^{na}). \quad (2.17)$$

Similarly, from

$$(-1)^{n-1-a} q^{\binom{n-a}{2}} \begin{bmatrix} -1-k \\ n-1-a \end{bmatrix}_q = (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} a-n \\ k \end{bmatrix}_q,$$

it follows that

$$\begin{aligned}
& (2.10) + (2.14) \\
& = s(-1)^a q^{\binom{a+1}{2}} \sum_{k=0}^{n-1-a} q^{k^2+k} \begin{bmatrix} n-1-a \\ k \end{bmatrix}_q \begin{bmatrix} a-n \\ k \end{bmatrix}_q f_k(q) \cdot ((-1)^{n-1} q^{\binom{n}{2}} - q^{nk}).
\end{aligned} \tag{2.18}$$

Noting that

$$1 + q^{\frac{n}{2}} = \frac{1 - q^n}{1 - q^{\frac{n}{2}}} \equiv 0 \pmod{\Phi_n(q)}$$

for those even n , we always have

$$(-1)^{n-1} q^{\binom{n}{2}} \equiv 1 \pmod{\Phi_n(q)}.$$

Thus by (2.15), (2.16), (2.17) and (2.18), we get

$$\begin{aligned}
& \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q \begin{bmatrix} -1-\alpha \\ k \end{bmatrix}_q ((-1)^a q^{\binom{a+1}{2}} f_k(q) - \hat{f}_k(q)) \\
& \equiv (-1)^a s q^{\binom{a+1}{2}} \sum_{k=0}^{n-1} q^{k^2+k} ((-1)^{n-1} q^{\binom{n}{2}} - q^{na}) \begin{bmatrix} -1-a \\ k \end{bmatrix}_q \begin{bmatrix} a \\ k \end{bmatrix}_q f_k(q) \\
& \equiv (-1)^a s q^{\binom{a+1}{2}} \sum_{k=0}^{n-1} q^{k^2+k} (1 - (-1)^{n-1} q^{na-\binom{n}{2}}) \begin{bmatrix} -1-\alpha \\ k \end{bmatrix}_q \begin{bmatrix} \alpha \\ k \end{bmatrix}_q f_k(q) \pmod{\Phi_n(q)^2}.
\end{aligned}$$

When n is odd, we have

$$s(1 - q^{na-s\binom{n}{2}}) \equiv 1 - q^{sna-s\binom{n}{2}} \pmod{\Phi_n(q)^2}.$$

So

$$\begin{aligned}
& (-1)^a q^{\binom{a+1}{2} + sna-s\binom{n}{2}} \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q \begin{bmatrix} -1-\alpha \\ k \end{bmatrix}_q f_k(q) \\
& \equiv \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q \begin{bmatrix} -1-\alpha \\ k \end{bmatrix}_q \hat{f}_k(q) \pmod{\Phi_n(q)^2}.
\end{aligned}$$

Suppose that n is even. Then

$$s(1 - (-1)^{n-1} q^{na-s\binom{n}{2}}) \equiv 1 - (-1)^s q^{sna-s\binom{n}{2}} \pmod{\Phi_n(q)^2}.$$

Then we have

$$\begin{aligned} & (-1)^{a+s} q^{\binom{a+1}{2} + sna - s \binom{n}{2}} \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q \begin{bmatrix} -1-\alpha \\ k \end{bmatrix}_q f_k(q) \\ & \equiv \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q \begin{bmatrix} -1-\alpha \\ k \end{bmatrix}_q \hat{f}_k(q) \pmod{\Phi_n(q)^2}. \end{aligned}$$

□

Remark. Let

$$f_k(q) = x^k.$$

Then

$$\hat{f}_k(q) = \sum_{j=0}^k (-1)^j q^{\binom{j+1}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q x^j = (xq; q)_k.$$

Then Theorem 1.1 implies a conjecture of Guo and Zeng [1, Conjecture 7.1].

Remark. Let

$$f_k(q) = \frac{q^k(x; q)_k}{(q; q)_k}.$$

Then it is not difficult to prove that

$$\tilde{f}_k(q) = \sum_{j=0}^k (-1)^j q^{\binom{j}{2} - kj} \begin{bmatrix} k \\ j \end{bmatrix}_q \cdot \frac{q^j(x; q)_j}{(q; q)_j} = \frac{(xq^{-1}; q^{-1})_k}{(q^{-1}; q^{-1})_k}.$$

Define

$$\mathcal{P}_n(\alpha, x; q) = \sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q \begin{bmatrix} -1-\alpha \\ k \end{bmatrix}_q \cdot \frac{(x; q)_k}{(q; q)_k}.$$

Then in view of Theorem 1.2, we can get

$$\mathcal{P}_n(-r/d, x; q^d) \equiv (-1)^a q^{d \binom{a+1}{2} + \frac{(ad+r)(n-1-2a)}{2}} \mathcal{P}_n(-r/d, xq^{-d}; q^{-d}) \pmod{\Phi_n(q)^2}$$

for odd $n \geq 3$, where $a = \langle -r/d \rangle_n$. This is a q -analogue of [2, Theorem 2.2].

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